Finite-Element Method with Edge Elements for Waveguides Loaded with Ferrite Magnetized in Arbitrary Direction

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Abstract—**In this paper, finite-element method (FEM) formulations with edge elements for directly calculating the phase constants of ferrite-loaded waveguides with arbitrarily magnetized directions are presented. Dispersion characteristics are calculated for the partially-filled ferrite rectangular waveguide structure, where the dc field is in any arbitrary direction including parallel to any of the three axes. The variation of phase constants with the direction of dc magnetic field is illustrated. In order to solve the quadratic eigenvalue equation, which appears in the case where the magnetized direction is not parallel to the propagation, a simple and effective approach is proposed with no increase in the size of the matrices.**

I. Introduction

IT IS well-known that traditional microwave guides loaded with ferrite materials are used in a number of microwave devices, such as phase shifters, resonators, circulators and tunable filters [1], [2]. Recent research has demonstrated that coupled planar waveguides and other structures loaded with ferrite layers magnetized parallel or perpendicular to the direction of wave propagation exhibit behavior, which might be used for novel microwave components at higher frequencies [3]–[6]. Therefore it is necessary to develop a method for the analysis and design of these kinds of devices with arbitrary magnetization direction.

The feature of ferrite materials is that the permeability is a tensor, which always depends on the operating frequency and applied dc magnetic field. This results in difficulties of analysis and calculation. Analytic solutions are known only for a few simple geometry structures, such as rectangular and circular waveguides with ferrite slab or rod, and the dc magnetic field is limited to the wave-propagation or basic coordinate-axis directions [7], [8]. For structures with arbitrarily magnetized ferrite numerical methods are necessary, and in any case a generally applicable approach is desirable.

The finite-element method (FEM) is an effective and accurate numerical method suitable for complex geometry and material properties, which has been widely used in this area. Konrad first proposed a three-component variational formulation for anisotropic media [9]. Its eigenvalues and eigenvectors correspond to the modal eigenfrequencies and three-component vector fields of waveguides. Wang and Ida successfully applied this formulation with edge elements to study the resonant frequencies of ferrite loaded cavities [10]. The so-called spurious mode interference, which usually occurs in FEM with vector variational formulation, has been eliminated by using edge elements or other approaches [11]. However, the formulation with frequencies as the eigenvalues is not convenient for solving waveguide problems. Therefore methods of directly calculating phase constants have appeared in the recent literature. Dillon, Gibson, and Webb applied FEM with covariant projection elements to the analysis of ferrite loaded waveguides, where the applied dc magnetic field is parallel to the propagation direction [12]. Anderson and Cendes published a vector finite solution of ferrite loaded waveguides, where the dc magnetic field is parallel or transverse to the direction of propagation [13]. Their method reduced the quadratic eigenvalue equation to a linear one, but not quite twice as large. It may also be pointed that Angkaew, Matsuhara, and Kumagai analyzed the waveguide with transversely magnetized ferrite early in 1987 [14]. But they used their own variational function and shape functions with four field components ($\mathbf{e}_t$ and $\mathbf{h}_t$) and more degrees of freedom than others.

In this paper, FEM formulas for both $\mathbf{E}$ and $\mathbf{H}$ fields with edge elements for directly calculating the phase constants of ferrite loaded waveguides with arbitrarily magnetized direction are presented for the first time. The dispersion characteristics are calculated for an example structure, where the dc field is arbitrarily directed as well as parallel to the three axes. In order to solve the quadratic eigenvalue equation, which appears in the case of the magnetized direction not parallel to the propagation, a simple and effective iteration approach is proposed with no increase in the size of the matrices.

II. Basic Formulations

A. Permeability Tensor of Arbitrarily-Magnetized Ferrite

Ferrite is a kind of anisotropic material, in which the magnetic induction $\mathbf{B}$ is not parallel to the magnetic field $\mathbf{H}$. The relationship between them can be expressed in terms of a permeability tensor, i.e.,

$$\mathbf{B} = \mu_o[\mu]\mathbf{H}. \tag{1}$$

Manuscript received February 20, 1995; revised February 15, 1996.

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Publisher Item Identifier S 0018-9480(96)03789-1.

When the applied dc magnetic field is parallel to the z-axis, the relative permeability tensor is given by

$$[\mu] = \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

with

$$\begin{aligned} \mu &= 1 + \chi \\ &= 1 + \frac{\mu_o \gamma^2 H_o M_s}{\mu_o^2 \gamma^2 H_o^2 - \omega^2} \end{aligned} \tag{2.1}$$

$$\kappa = \frac{\gamma \omega M_s}{\mu_o^2 \gamma^2 H_o^2 - \omega^2} \tag{2.2}$$

where M_s and γ have their usual meanings [7].

Now assuming the applied dc field $\mathbf{H}_o$ is directed along an arbitrary direction, i.e., $\langle \mathbf{H}_o, \hat{\mathbf{z}} \rangle = \theta$, $\langle \mathbf{H}_{oxy}, \hat{\mathbf{x}} \rangle = \varphi$ as shown in Fig. 1, it can be proved by using coordinate transformation that $\mathbf{B} = \mu_o [\mu] \mathbf{H}$ remains valid, but $[\mu]$ become a full matrix of the form:

$$[\mu] = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \tag{3}$$

where

$$\mu_{11} = \mu(\cos^2\theta \cdot \cos^2\varphi + \sin^2\varphi) + \sin^2\theta \cdot \cos^2\varphi \tag{3.1}$$
$$\mu_{12} = (1-\mu)\sin^2\theta \cdot \sin\varphi \cdot \cos\varphi - j\kappa\cos\theta \tag{3.2}$$
$$\mu_{13} = (1-\mu)\sin\theta \cdot \cos\theta \cdot \cos\varphi + j\kappa\sin\theta \cdot \sin\varphi \tag{3.3}$$
$$\mu_{21} = (1-\mu)\sin^2\theta \cdot \sin\varphi \cdot \cos\varphi + j\kappa\cos\theta \tag{3.4}$$
$$\mu_{22} = \mu(\cos^2\theta \cdot \sin^2\varphi + \cos^2\varphi) + \sin^2\theta \cdot \sin^2\varphi \tag{3.5}$$
$$\mu_{23} = (1-\mu)\sin\theta \cdot \cos\theta \cdot \sin\varphi - j\kappa\sin\theta \cdot \cos\varphi \tag{3.6}$$
$$\mu_{31} = (1-\mu)\sin\theta \cdot \cos\theta \cdot \cos\varphi - j\kappa\sin\theta \cdot \sin\varphi \tag{3.7}$$
$$\mu_{32} = (1-\mu)\sin\theta \cdot \cos\theta \cdot \sin\varphi + j\kappa\sin\theta \cdot \cos\varphi \tag{3.8}$$
$$\mu_{33} = \mu\sin^2\theta + \cos^2\theta. \tag{3.9}$$

Obviously, the above expression is in agreement with (2) when $\mathbf{H}_o \parallel \hat{\mathbf{z}}$, i.e., $\theta = 0°$. When $\mathbf{H}_o$ is in the xy plane, i.e., $\theta = 90°$, and when $\mathbf{H}_o$ is in the xz plane, i.e., $\varphi = 0°$ it has the forms, respectively

$$[\mu] = \begin{bmatrix} \mu\sin^2\varphi + \cos^2\varphi & (1-\mu)\sin\varphi \cdot \cos\varphi & j\kappa \cdot \sin\varphi \\ (1-\mu)\sin\varphi \cdot \cos\varphi & \mu\cos^2\varphi + \sin^2\varphi & -j\kappa \cdot \cos\varphi \\ -j\kappa \cdot \sin\varphi & j\kappa \cdot \cos\varphi & \mu \end{bmatrix} \tag{4.1}$$

$$[\mu] = \begin{bmatrix} \mu\cos^2\theta + \sin^2\theta & -j\kappa \cdot \cos\theta & (1-\mu)\sin\theta \cdot \cos\theta \\ j\kappa \cdot \cos\theta & \mu & -j\kappa \cdot \sin\theta \\ (1-\mu)\sin\theta \cdot \cos\theta & j\kappa \cdot \sin\theta & \mu\sin^2\theta + \cos^2\theta \end{bmatrix}. \tag{4.2}$$

Equation (4.2) is identical to (2.51) in [2]. It should be pointed that the permeability tensor (3), generally speaking, is full and complex, and the inverse matrix also is full and complex.

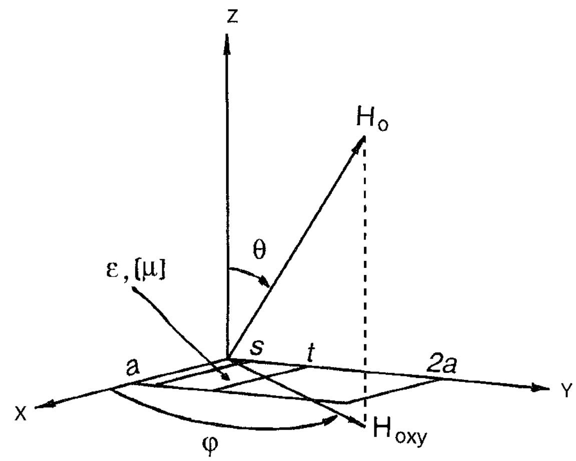

Fig. 1. Partially-filled ferrite rectangular waveguide cross section with applied dc magnetic field in an arbitrary direction.

B. Finite-Element Formulas with Edge Elements

We consider a waveguide loaded with ferrite of the relative permeability tensor of the (3) and assume that the EM fields in the waveguide vary as $\exp[j(\omega t - \beta z)]$. The applied magnetic field is assumed to be along an arbitrary space direction. From Maxwell equations, the following vectorial wave equation and corresponding variational function can be obtained in the form

$$\nabla \times ([A] \nabla \times \mathbf{G}) - k_o^2 [B] \mathbf{G} = 0 \tag{5}$$

and

$$\begin{aligned} F(\mathbf{G}) = \iint_{(s)} \{ &(\nabla \times \mathbf{G})^* [A] (\nabla \times \mathbf{G}) \\ &- k_o^2 \mathbf{G}^* [B] \mathbf{G} \} \, dx \, dy \end{aligned} \tag{6}$$

where $k_o = \omega / c$ is the free-space wave number, s is the waveguide cross section and the asterisk denotes complex conjugate, and

$$\begin{aligned} [A] &= \varepsilon^{-1}[I], \\ [B] &= [\mu] \quad \text{for} \quad \mathbf{G} = \mathbf{H} \\ [A] &= [\mu]^{-1}, \\ [B] &= \varepsilon[I] \quad \text{for} \quad \mathbf{G} = \mathbf{E}. \end{aligned} \tag{6.1}$$

In order to discretize the integral equation (6), the cross section of the waveguide is divided into a finite number of triangles. In each triangle the material properties are constant and the magnetic or electric field $\mathbf{G}(x, y, z)$ can be expressed in terms of the field values at the vertices and edges, i.e.,

$$\begin{aligned} \mathbf{G}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = &[G_x(x, y)\hat{\mathbf{x}} + G_y(x, y)\hat{\mathbf{y}} + G_z(x, y)\hat{\mathbf{z}}] \\ &\cdot \exp(-j\beta z) \end{aligned} \tag{7}$$

with

$$G_x(x, y) = \sum_{i=1}^{3} g_{ti} W_{xi}(x, y) \tag{7.1}$$

$$G_y(x, y) = \sum_{i=1}^{3} g_{ti} W_{yi}(x, y) \tag{7.2}$$

$$G_z(x, y) = \sum_{i=1}^{3} g_{zi} j\lambda_i(x, y) \qquad (7.3)$$

where λ_i is the scalar shape function and defined in terms of the Cartesian coordinates

$$[\lambda]^T = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} d_1 & d_{1x} & d_{1y} \\ d_2 & d_{2x} & d_{2y} \\ d_3 & d_{3x} & d_{3y} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \qquad (8)$$

where

$$\begin{bmatrix} d_i \\ d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} \dfrac{x_j y_k - x_k y_j}{2s} \\ \dfrac{y_j - y_k}{2s} \\ \dfrac{x_k - x_j}{2s} \end{bmatrix} \qquad (8.1)$$

$$2s = \det\left(\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right) \qquad (8.2)$$

where $(i, j, k) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$.

$\mathbf{W}_i(x, y) = W_{ix}(x, y)\hat{\mathbf{x}} + W_{iy}(x, y)\hat{\mathbf{y}}$ is the vectorial shape function corresponding to the ith edge and defined in terms of the scalar shape functions which are related to the two ends of the edge

$$\mathbf{W}_i(x, y) = \lambda_{(i1)}(x, y) \nabla \lambda_{(i2)}(x, y) - \lambda_{(i2)}(x, y) \nabla \lambda_{(i1)}(x, y) \qquad (9)$$

where, ∇ is a gradient operator. The relationship between the number of vertices and the ends of edges are shown in Fig. 2. g_{zi} is the z-component of the field G at the ith vertex. g_{ti} is the linear integral of the field along the ith edge. Components of the vector $\mathbf{W}_i(x, y)$ can be written in matrix form, as follows

$$[W_x]^T = \begin{bmatrix} W_{1x} \\ W_{2x} \\ W_{3x} \end{bmatrix} = \begin{bmatrix} d_{2x}\lambda_1 - d_{1x}\lambda_2 \\ d_{3x}\lambda_2 - d_{2x}\lambda_3 \\ d_{1x}\lambda_3 - d_{3x}\lambda_1 \end{bmatrix} = \begin{bmatrix} a_{1x} - b_1 y \\ a_{2x} - b_2 y \\ a_{3x} - b_3 y \end{bmatrix} \qquad (10.1)$$

$$[W_y]^T = \begin{bmatrix} W_{1y} \\ W_{2y} \\ W_{3y} \end{bmatrix} = \begin{bmatrix} d_{2y}\lambda_1 - d_{1y}\lambda_2 \\ d_{3y}\lambda_2 - d_{2y}\lambda_3 \\ d_{1y}\lambda_3 - d_{3y}\lambda_1 \end{bmatrix}$$

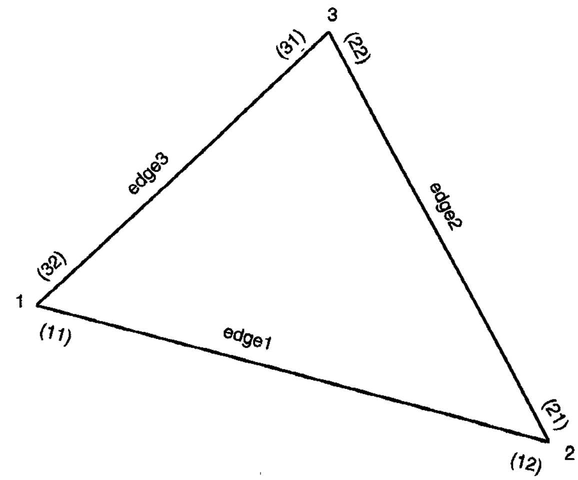

Fig. 2. The relation between numbers of vertices and ends of edges for a triangle element.

$$= \begin{bmatrix} a_{1y} + b_1 x \\ a_{2y} + b_2 x \\ a_{3y} + b_3 x \end{bmatrix} \qquad (10.2)$$

$$\begin{bmatrix} a_{ix} \\ a_{iy} \\ b_i \end{bmatrix} = \begin{bmatrix} d_i d_{jx} - d_j d_{ix} \\ d_i d_{jy} - d_j d_{iy} \\ d_{ix} d_{jy} - d_{jx} d_{iy} \end{bmatrix} \qquad (10.3)$$

where $(i, j) = (1, 2)$ or $(2, 3)$ or $(3, 1)$. From (7)–(10), one can also obtain

$$\left[\frac{\partial W_y}{\partial x}\right] = -\left[\frac{\partial W_x}{\partial y}\right] = [b_1, b_2, b_3] \qquad (11.1)$$

$$\left[\frac{\partial \lambda}{\partial x}\right] = [d_{1x}, d_{2x}, d_{3x}] \qquad (11.2)$$

$$\left[\frac{\partial \lambda}{\partial y}\right] = [d_{1y}, d_{2y}, d_{3y}]. \qquad (11.3)$$

Equations (7)–(7.3) can also be rewritten as the matrix form

$$\mathbf{G} = \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} \exp(-j\beta z) = \begin{bmatrix} [W_x] & [0] \\ [W_y] & [0] \\ [0] & j[\lambda] \end{bmatrix} \begin{bmatrix} [g_t] \\ [g_z] \end{bmatrix} \exp(-j\beta z) \qquad (12)$$

with

$$[g_t] = [g_{t1}, g_{t2}, g_{t3}]^T \qquad (12.1)$$

$$[g_z] = [g_{z1}, g_{z2}, g_{z3}]^T. \qquad (12.2)$$

Taking the curl of (12), we obtain

$$\nabla \times \mathbf{G} = \begin{bmatrix} j\beta[W_y] & j\left[\dfrac{\partial \lambda}{\partial y}\right] \\ -j\beta[W_x] & -j\left[\dfrac{\partial \lambda}{\partial x}\right] \\ 2\left[\dfrac{\partial W_y}{\partial x}\right] & [0] \end{bmatrix} \cdot \begin{bmatrix} [g_t] \\ [g_z] \end{bmatrix} \exp(-j\beta z). \qquad (13)$$

Substituting (12) and (13) into the variational function (6), using the Rayleigh–Ritz procedure, i.e., making it stationary with respect to all the independent variables g_t and g_z, the variational function (6) in the integral form can be discretized and transformed to a matrix eigenvalue equation with the frequency squared as the eigenvalue

$$\begin{bmatrix} \beta^2 A_{tt}^{(1)} + \beta A_{tt}^{(2)} + A_{tt}^{(3)} & \beta A_{tz}^{(1)} + A_{tz}^{(2)} \\ \beta A_{zt}^{(1)} + A_{zt}^{(2)} & A_{zz} \end{bmatrix} \begin{bmatrix} [g_t] \\ [g_z] \end{bmatrix} = k_o^2 \begin{bmatrix} B_{tt} & B_{tz} \\ B_{zt} & B_{zz} \end{bmatrix} \begin{bmatrix} [g_t] \\ [g_z] \end{bmatrix} \tag{14}$$

where $[g_t]$ and $[g_z]$ are column vectors, which consist of all the independent unknowns g_{ti} and g_{zi} of the cross section; the matrices are sparse but not symmetric, and their elements are defined as follows:

$$A_{tt}^{(1)} = \sum_{(e)} \iint \{[W_y]^T [W_y] a_{11} + [W_x]^T [W_x] a_{22} - [W_y]^T [W_x] a_{12} - [W_x]^T [W_y] a_{21}\} \, dx \, dy \tag{15.1}$$

$$A_{tt}^{(2)} = \sum_{(e)} \iint 2j \left\{ \left[\frac{\partial W_y}{\partial x}\right]^T [W_y] a_{31} - \left[\frac{\partial W_y}{\partial x}\right]^T [W_x] a_{32} - [W_y]^T \left[\frac{\partial W_y}{\partial x}\right] a_{13} + [W_x]^T \left[\frac{\partial W_y}{\partial x}\right] a_{23} \right\} dx \, dy \tag{15.2}$$

$$A_{tt}^{(3)} = \sum_{(e)} \iint 4 \left\{ \left[\frac{\partial W_y}{\partial x}\right]^T \left[\frac{\partial W_y}{\partial x}\right] a_{33} \right\} dx \, dy \tag{15.3}$$

$$A_{tz}^{(1)} = \sum_{(e)} \iint \left\{ [W_y]^T \left[\frac{\partial \lambda}{\partial y}\right] a_{11} + [W_x]^T \left[\frac{\partial \lambda}{\partial x}\right] a_{22} - [W_x]^T \left[\frac{\partial \lambda}{\partial y}\right] a_{21} - [W_y]^T \left[\frac{\partial \lambda}{\partial x}\right] a_{12} \right\} dx \, dy \tag{15.4}$$

$$A_{tz}^{(2)} = \sum_{(e)} \iint 2j \left\{ \left[\frac{\partial W_y}{\partial x}\right]^T \left[\frac{\partial \lambda}{\partial y}\right] a_{31} - \left[\frac{\partial W_y}{\partial x}\right]^T \left[\frac{\partial \lambda}{\partial x}\right] a_{32} \right\} dx \, dy \tag{15.5}$$

$$A_{zt}^{(1)} = \sum_{(e)} \iint \left\{ \left[\frac{\partial \lambda}{\partial y}\right]^T [W_y] a_{11} + \left[\frac{\partial \lambda}{\partial x}\right]^T [W_x] a_{22} - \left[\frac{\partial \lambda}{\partial x}\right]^T [W_y] a_{21} - \left[\frac{\partial \lambda}{\partial y}\right]^T [W_y] a_{12} \right\} dx \, dy \tag{15.6}$$

$$A_{zt}^{(2)} = \sum_{(e)} \iint 2j \left\{ - \left[\frac{\partial \lambda}{\partial y}\right]^T \left[\frac{\partial W_y}{\partial x}\right] a_{13} + \left[\frac{\partial \lambda}{\partial x}\right]^T \left[\frac{\partial W_y}{\partial x}\right] a_{23} \right\} dx \, dy \tag{15.7}$$

$$A_{zz} = \sum_{(e)} \iint \left\{ \left[\frac{\partial \lambda}{\partial y}\right]^T \left[\frac{\partial \lambda}{\partial y}\right] a_{11} + \left[\frac{\partial \lambda}{\partial x}\right]^T \left[\frac{\partial \lambda}{\partial x}\right] a_{22} - \left[\frac{\partial \lambda}{\partial x}\right]^T \left[\frac{\partial \lambda}{\partial y}\right] a_{21} - \left[\frac{\partial \lambda}{\partial y}\right]^T \left[\frac{\partial \lambda}{\partial x}\right] a_{12} \right\} dx \, dy \tag{15.8}$$

$$B_{tt} = \sum_{(e)} \iint \{[W_x]^T [W_x] b_{11} + [W_y]^T [W_y] b_{22} + [W_y]^T [W_x] b_{21} + [W_x]^T [W_y] b_{12}\} \, dx \, dy \tag{15.9}$$

$$B_{tz} = \sum_{(e)} \iint j\{[W_x]^T [\lambda] b_{13} + [W_y]^T [\lambda] b_{23}\} \, dx \, dy \tag{15.10}$$

$$B_{zt} = \sum_{(e)} \iint -j\{[\lambda]^T [W_x] b_{31} + [\lambda]^T [W_y] b_{32}\} \, dx \, dy \tag{15.11}$$

$$B_{zz} = \sum_{(e)} \iint \{[\lambda]^T [\lambda] b_{33}\} \, dx \, dy \tag{15.12}$$

where a_{ij} and b_{ij} are elements of matrices $[A]$ and $[B]$ in (6), and the integration results are easily obtained.

III. Approaches for Solving Eigenvalue Equations

A. Calculation of Cut-off Frequencies of Waveguides

In the case of cut-off, i.e., the phase constant $\beta = 0$, (14) takes the form of

$$\begin{bmatrix} A_{tt}^{(3)} & A_{tz}^{(2)} \\ A_{zt}^{(2)} & A_{zz} \end{bmatrix} \begin{bmatrix} g_t \\ [g_z] \end{bmatrix} = k_o^2 \begin{bmatrix} B_{tt} & B_{tz} \\ B_{zt} & B_{zz} \end{bmatrix} \begin{bmatrix} [g_t] \\ [g_z] \end{bmatrix} \tag{16}$$

which is the eigenvalue equation with cut-off frequencies as the eigenvalues. For general materials, the properties of which are independent of frequency, solving the equation is simple and straightforward. However, it is not easy to solve it for the ferrite loaded waveguides because, as mentioned above, the permeability tensor is dependent not only on the applied dc magnetic field but also on the operating frequency. This means that the matrix elements of (16) are functions of k_o^2. Therefore, (16) is a transcendental eigenvalue equation. A commonly used method for solving this equation is first to compute the frequency at a specified value of (κ/μ), and then to calculate the applied field based on the calculated frequency and specified value of (κ/μ). This procedure is opposite to the practical application. The practical approach would be to solve for frequencies at the specified applied fields. In order to follow the more intuitively practical approach we proposed a iteration technique, which has been successfully used in FEM solutions for nonlinear optical waveguides and electrical resonant cavities [15], [16].

Starting with $[\mu] = \mu[I]$ (the value of the permeability tensor when dc magnetic field does not exist), one can get the eigen frequency $\omega_{(i)} = k_{o(i)}c$ of the ith $(i = 1)$ step by using (16); then one can obtain the permeability tensor of the ith step, $[\mu_{(i)}]$ by using (3) or (4); after substituting it into (16), one can get the frequency of the $(i+1)$th step, $\omega_{(i+1)} = k_{o(i+1)}c$. Iteration continues until the difference between eigenvalues of the two contiguous steps is less than a given tolerance.

B. Calculation of Phase Constants of Waveguides

1) $\mathbf{H}_o$ *Parallel to the Propagation Direction:* When the applied dc magnetic field is parallel to the wave propagation, i.e., the $\hat{\mathbf{z}}$ direction, according to (2), (6), and (15), both for the electric formulation [i.e., $\mathbf{G} = \mathbf{E}$ in (16)] and magnetic formulation [i.e., $\mathbf{G} = \mathbf{H}$ in (6)], we have

$$A_{tt}^{(2)} = A_{tz}^{(2)} = A_{zt}^{(2)} = 0 \tag{17}$$

and

$$B_{tz} = B_{zt} = 0. \tag{18}$$

Making a transformation of variables

$$[g_z] = \beta[\tilde{g}_z]. \tag{19}$$

Equation (16) becomes

$$\begin{bmatrix} B_{tt} - \dfrac{A_{tt}^{(3)}}{k_o^2} & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [g_t] \\ [\tilde{g}_z] \end{bmatrix} = \left(\frac{\beta}{k_o}\right)^2 \begin{bmatrix} A_{tt}^{(1)} & A_{tz}^{(1)} \\ A_{zt}^{(1)} & A_{zz} - B_{zz}k_o^2 \end{bmatrix} \begin{bmatrix} [g_t] \\ [\tilde{g}_z] \end{bmatrix} \tag{20}$$

Equation (20) is an eigen equation both for electric and magnetic fields with $(\beta/k_o)^2$ as the eigenvalues, by which it is very convenient to calculate directly phase constants based on specified frequencies $k_o = \omega/c$, applied magnetic field H_o and magnetization M_s.

2) Arbitrary-Direction Applied DC Magnetic Field: For an arbitrary direction of the dc magnetic field, (17) and (18) are no longer simultaneously valid [i.e., we have (17) for the $\mathbf{H}$ formulation and we have (18) for the $\mathbf{E}$ formulation]. Generally, instead of (20), we obtain

$$\begin{bmatrix} B_{tt} - \dfrac{A_{tt}^{(3)}}{k_o^2} & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [g_t] \\ [\tilde{g}_z] \end{bmatrix} + \Delta(\beta) = \left(\frac{\beta}{k_o}\right)^2 \begin{bmatrix} A_{tt}^{(1)} & A_{tz}^{(1)} \\ A_{zt}^{(1)} & A_{zz} - B_{zz}k_o^2 \end{bmatrix} \begin{bmatrix} [g_t] \\ [\tilde{g}_z] \end{bmatrix} \tag{21}$$

where

$$\Delta(\beta) = -\frac{\beta}{k_o^2} \begin{bmatrix} A_{tt}^{(2)} & (A_{tz}^{(2)} & -B_{tz}k_o^2) \\ (A_{zt}^{(2)} & -B_{zt}k_o^2) & [0] \end{bmatrix} \begin{bmatrix} g_t \\ [\tilde{g}_z] \end{bmatrix}. \tag{21.1}$$

Obviously, (21) is a quadratic eigenequation, which contains both β and β^2. The method widely used for solving this problem is to reduce this quadratic equation to a linear one with the matrix order twice the original. However there is a double penalty to pay for this transformation: it increases both the computing time and the memory requirement, because doubling the matrix order usually causes an increase of $2^3 (= 8)$ in CPU time for general eigenvalue solvers. Here an iteration method is proposed, in which $\Delta(\beta)$ in (21) is simply used as improving term. The iteration is straightforward without any transformation, and calculation shows that the average number of iterations is much less than eight when drawing a dispersion characteric curve. The procedure is as follows: substituting a reasonable initial β (for instance $\beta/k_o = 0$ or 1) used as the $\beta_{(i)}$ of the ith step ($i = 0$) into $\Delta(\beta)$ of (21), solving the eigenvalue equation with $(\beta/k_o)^2$ as the eigenvalues, one can get $\beta_{(i+1)}$ of the $(i+1)$th step. Iteration does not stop until the difference between $(\beta/k_o)^2$ values of two contiguous steps is less than a given tolerance.

IV. Numerical Results and Discussion

A. Partially-Filled Ferrite Rectangular Waveguide

As a numerical example, we consider the well-known partially-filled ferrite rectangular waveguide, shown in Fig. 1. Ferrite-loaded waveguides are usually used as nonreciprocal microwave components, in which the ferrite slab or rod is asymmetrically placed in the waveguide. In order to compare our results with the published analytical data, the geometry and parameters are chosen to be exactly the same as those of [7], [14]: $s = a/4$, $t = 3a/4$, $\varepsilon_f = 10$, $\gamma M_s a/c = 0.5$, $\gamma\mu_o H_o a/c = 0.5$. In order to generalize the calculated results we normalize all qualities related to length by using the short side of the waveguide, a, and introduce the normalized frequency $\overline{k}_o = k_o a = \omega a/c$. The practical frequency $\omega = \overline{k}_o c/a$. In the calculation there are 104 triangle elements and 175 degrees of freedom. Calculation shows no spurious modes for the frequency-eigenvalue equation, all solutions are physical except a certain number of zero-eigenvalues, and the smallest nonzero solution corresponds to the fundamental mode. For the phase-constant eigenvalue equation, all the positive solutions are related to the propagation modes, and the largest eigenvalue corresponds to the $(\beta/k_o)^2$ of the fundamental modes.

Fig. 3 presents the dispersion characteristics of the three lowest modes for wave propagation in both the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions. The applied dc magnetic field is parallel to the $+\hat{\mathbf{x}}$ direction. The corresponding analytical solutions are presented by circles and asterisks. As shown in Fig. 3, our results are in good agreement with the analytical solutions at lower frequencies for each modes, but a little different at frequencies much higher than the cut-off frequencies because of an insufficient number of elements. Our results are very close to ones shown in Fig. 7 of [14], but the number of degrees of freedom used here is about half that used in [14].

Figs. 4 and 5 illustrate the variation of phase constants with the direction of dc field $\mathbf{H}_o$. In Fig. 4 the direction of $\mathbf{H}_o$ varies from $+\hat{\mathbf{x}}$ to $-\hat{\mathbf{x}}$ in the xy plane ($\theta = 90°$). It can be seen that, as expected, the strongest nonreciprocity occurs when the field is parallel with the x-axis ($\varphi = 0°$ and $\varphi = 180°$) and complete reciprocity when the field is parallel with the y-axis ($\varphi = 90°$). For example, the difference between β^+/k_o and β^-/k_o is about 0.6 near the x-axis but they are equal near the y-axis for $\overline{k}_o = 1.1$. Fig. 5 presents variation of the phase constants when the direction of $\mathbf{H}_o$ varies from $+\hat{\mathbf{z}}$ to

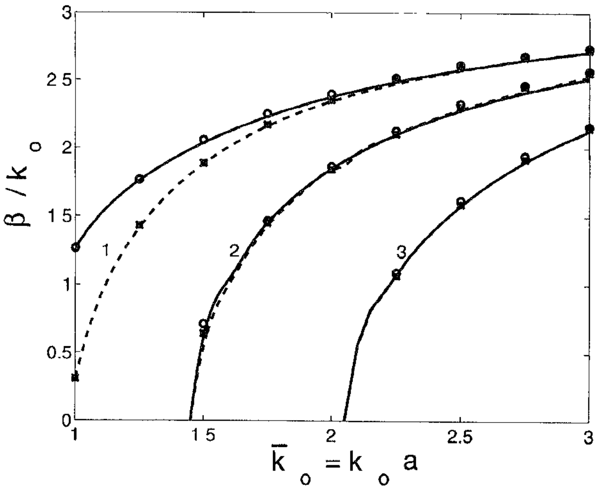

Fig. 3. Dispersion characteristics for the waveguide shown in Fig. 1 with $\theta = 90°$ and $\varphi = 0°$. Solid line for β^+, dashed line for β^-. 1, 2, and 3 for the fundamental, second and third modes, respectively.

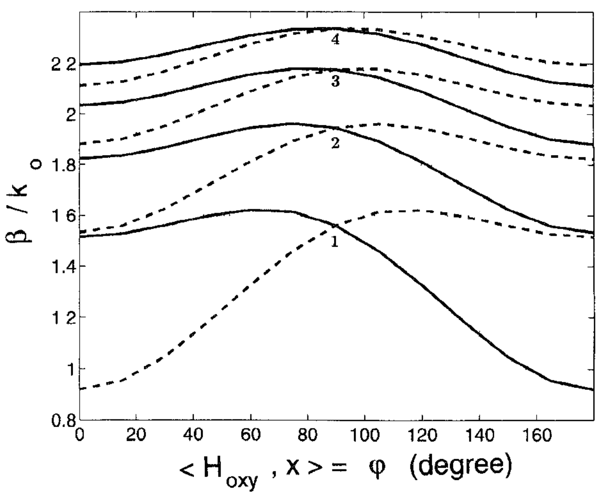

Fig 4. Variation of phase constants of the fundamental mode with the direction (φ) of the applied dc magnetic field in the xy plane ($\theta = rm90°$). Solid line for β^+, dashed line for β^-. 1, 2, 3, and 4 for $\bar{k}_o$ = 1.1, 1.3, 1.5, and 1.7, respectively

$-\hat{\mathbf{z}}$ in the xz plane ($\varphi = 0°$). It can be seen again that the strongest nonreciprocity occurs with $\mathbf{H}_o$ along the x-axis and complete reciprocity with $\mathbf{H}_o$ along the z-axis. Therefore, one can control wave propagation by changing the direction of the dc field $\mathbf{H}_o$, and by its strength.

B. Discussion on Convergency

In this paper, iteration approaches are proposed for solving the linear eigenvalue equation for cut-off frequencies and the quadratic eigenvalue equation for phase constants with an arbitrarily-directed dc magnetic field. Here we discuss the convergency of the latter. From the physical point of view, any frequency has the corresponding phase constants ($\beta^2 > 0$ for propagation modes or $\beta^2 \leq 0$ for cut-off and evanescent modes). From the mathematical viewpoint, the improving term $\Delta(\beta)$ in (21) originates as a part of the equation and is not artificially added. Therefore (21) is naturally convergent. This has been proved by calculation.

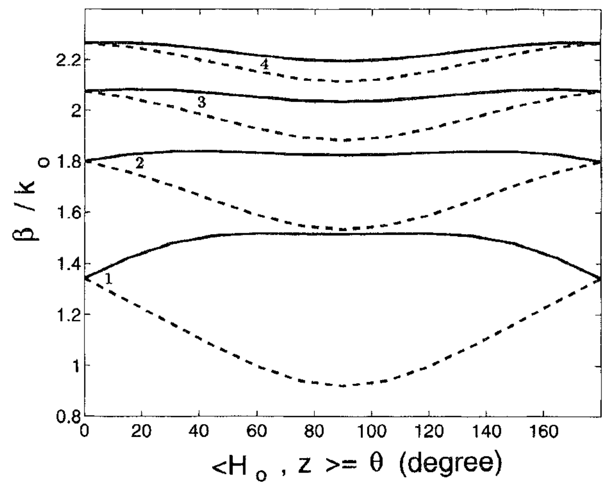

Fig. 5. Variation of phase constants of the fundamental mode with the direction (θ) of the applied dc magnetic field in the xz plane ($\varphi = 0°$). Solid line for β^+, dashed line for β^-. 1, 2, 3, and 4 for $\bar{k}_o$ = 1.1, 1.3, 1.5, and 1.7, respectively.

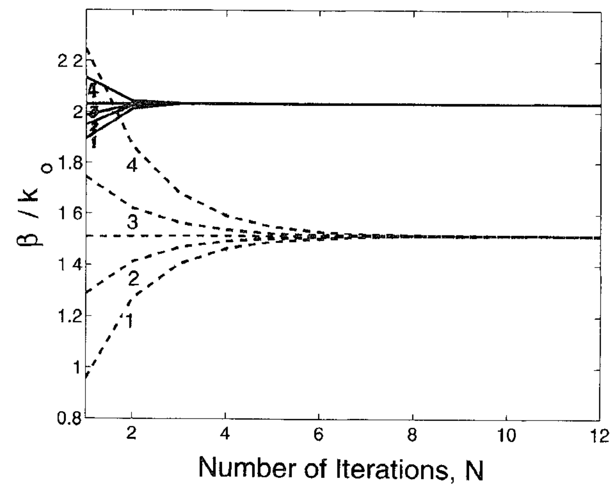

Fig. 6. Influence of the initial value $(\beta/k_o)_{\text{initial}}$ on convergency. Solid line for $\bar{k}_o$ = 1 5, dashed line for $\bar{k}_o$ = 1.1, both for the fundamental mode. 1: $(\beta/k_o)_{\text{initial}}$ = 0.0, 2: $(\beta/k_o)_{\text{initial}}$ = 1 0, 3: $(\beta/k_o)_{\text{initial}}$ = 2.0, 4: $(\beta/k_o)_{\text{initial}}$ = 3.0.

Of course, convergent speed, i.e., the number of iterations required, depends on the initial β value and the required tolerance as well as the mode and frequency. Fig. 6 shows the number of iterations, N, with different initial β values for the first mode and two specified frequencies $\bar{k}_o = 1.1$ and 1.5, from which it can be seen that fewer iterations are required:

- when the frequency is well above cut-off;
- the closer the initial β is to the final value; and
- the larger the tolerance is.

In our calculation the initial β/k_o is set to be 1.0, the tolerance is set to be 10^{-4}, the number of iterations is usually less than 8. The number of iterations will decrease by 1/4–1/3 if the tolerance is increased up to 10^{-3}. It should be pointed that when a set of data is computed for drawing a continuous curve, if the calculated β of a point is used as the initial β of the next point, calculation shows that, the average iteration time will be sharply reduced. Table I presents the number of

TABLE I
NUMBERS OF ITERATIONS AND AVERAGE VALUES FOR THREE CURVES

Point No.	1	2	3	4	5	6	7	8	9	10	11	12	13	Average
Curve 1 in Fig.3	10	5	3	3	2	2	2	2	2	1				3.2
Curve 1 in Fig.4	4	2	2	2	2	2	1	2	2	2	2	2	2	2.1
Curve 1 in Fig.5	1	1	1	2	2	2	2	2	2	2	2	1	1	1.6

iterations and the average values for drawing the three curves in Figs. 3–5, respectively, from which it can be seen that the average number of iterations is reduced to 2–3.

It is well-known that transforming a quadratic eigenvalue problem to a linear one (if it is possible) always causes the matrix order to double, which always results in an 8 times increase in CPU time for general eigenvalue solvers. That doubles the cost of the method with not only more memory but also more computing time. The iteration approach proposed in this paper needs much less CPU time and no additional memory. Therefore this is a more economic and effective method.

Finally it is pointed out that it is not necessary to develop two computer codes separately for (20) and (21), because the latter includes the former in the case of the dc magnetic field in the propagation direction. In this circumstance $\Delta(\beta) = 0$ and no iterations are required. This is can be seen from the third set of data in Table I.

V. CONCLUSION

In this paper, FEM formulations with edge elements for the analysis of ferrite loaded waveguides with an arbitrarily-directed dc magnetic field are presented. The formulation for the case where the dc field is parallel to the direction of propagation is direct and convenient. In order to deal with the frequency dependence of the permeability tensor in the frequency-eigenvalue equation and the quadratic eigenequation of phase constants for arbitrarily-directed static magnetic field, the iteration approaches are proposed. Calculation for the classic waveguide example proved the accuracy and efficiency of the formulation and approach.

ACKNOWLEDGMENT

The authors would like to thank C. S. Teoh for helpful discussions.

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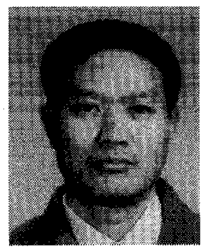

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